JOURNAL OF APPROXIMATION THEORY 48, 181–188 (1986)

An Estimate for the Coefficients of Polynomials of Given Length

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Communicated by Paul G. Nevai

Received February 25, 1985

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We use the following notation: P is a Dirichlet polynomial i.e.,

$$P(x)=\sum_{k=1}^{m} a_{n_k} x^{n_k} \tag{1.1}$$

where n_k, $k=1, 2,..., m$, are *real* numbers, $0 \leqslant n_1 < n_2 < \cdots < n_m$; $a_n(P)$ is the coefficient of x^n in $P(x)$, and the length of P is the number of non-zero coefficients in P. Further, p and q are always conjugate indices, i.e., $1 \leqslant p \leqslant +\infty$ and $1/p+1/q=1$. The interval over which the L_p-norm is taken is assumed, without loss of generality, to be of the form $[c, 1]$, $0 \leqslant c < 1$. (Since the length of a polynomial is not translation invariant, we cannot reduce all considerations to the case $c=0$.)

In [1] the following lemma played a crucial role: For every $r \in (0, 1)$ and for every polynomial P of length $\leqslant m$

$$|a_n(P)| \leqslant \frac{K}{r^n} \|P\|_{L_p(c,1)} \tag{1.2}$$

where $K=K(c, p; m, r)$ depends neither on n nor on the polynomial P.

In this paper we give the following improvement of (1.2):

THEOREM 1. *If P is a Dirichlet polynomial of length m, given by* (1.1) *and if*

$$|n_k - n_s| \geqslant 1 \qquad \text{for} \quad k=1, 2,..., m, \quad k \neq s \tag{1.3}$$

0021-9045/86 $3.00

then

$$|a_{n_s}(P)| \leqslant C(n_s+1)^{m-1/q} \|P\|_{L_p(c,1)} \tag{1.4}$$

where $C = C(c, m)$ depends neither on n_s nor on P.

The proof gives some information about the constant C. Let $A = (2e/(1-c^{1/(m-1)}))^{m-1}$. Then (1.4) holds with $C = A$ if $p = \infty$, and with $C = 2^m A(1+A)$ if $p < \infty$. The estimate (1.4) is the best possible in the sense that there exists $B_m > 0$ and polynomials $P_{m,n}$, $n = 1, 2,...$ of length m such that

$$|a_n(P_{m,n})| \geqslant B_m(n+1)^{m-1/q} \|P_{m,n}\|_{L_p(c,1)} \tag{1.5}$$

(see [3]). It is sufficient to let $P_{m,n} = x^n(1-x)^{m-1}$. Then $a_n(P) = 1$ and it is easy to check that

$$\|P_{m,n}\|_{L_p(0,1)} \leqslant \frac{K_m}{(n+1)^{m-1/q}},$$

so that (1.5) holds with $B_m = 1/K_m$.

Instead of (3) we can consider the apparently more general condition

$$\operatorname*{Min}_{k, k \neq s} |n_k - n_s| = h. \tag{1.6}$$

If the exponents n_k in (1.1) satisfy (1.6) with $h \neq 1$ let

$$\tilde{P}(x) = \sum_{k=1}^{m} \tilde{a}_{\nu_k} x^{\nu_k}, \qquad \text{where} \quad \tilde{a}_{\nu_k} = a_{n_k}$$

and $\nu_k = 1/h(n_k + (1-h)/p)$. Since

$$a_{n_s}(P) = \tilde{a}_{\nu_s}(\tilde{P}), \qquad |\nu_k - \nu_s| \geqslant 1$$

and

$$\|\tilde{P}\|_{L_p(c^h,1)} = h^{1/p} \|P\|_{L_p(c,1)},$$

applying Theorem 1 to $\tilde{P}$ we obtain the following result: If P is a Dirichlet polynomial of length m, given by (1.1), if (1.6) holds, and if $h \geqslant h_0 > 0$, then

$$|a_{n_s}(P)| \leqslant C\left(n_s + \frac{1}{p} + \frac{h}{q}\right)^{m-1/q} h^{1-m} \|P\|_{L_p(c,1)},$$

where $C = C(c, m, h_0)$ is independent of n_s, h, and P.

From

$$(1/|a_{n_s}|) \left\|\sum a_{n_k} x^{n_k}\right\|_{L_p(c,1)} = \left\|x^{n_s} - \sum_{k \neq s} b_{n_k} x^{n_k}\right\|_{L_p(c,1)} \tag{1.7}$$

where $b_{n_k} = -a_{n_k}/a_{n_s}$, it follows that any upper bound for the coefficient $|a_{n_s}|$ provides a lower bound for the right-hand side of (1.7). Thus Theorem 1 can be phrased in the following dual form:

THEOREM 1′. *Let* $\lambda \geqslant 0$ *and*

$$d_{c,p}(x^\lambda, N) = \inf\left\{\left\|x^\lambda - \sum_{j=1}^{N} b_j x^{\lambda_j}\right\|_{L_p(c,1)} : b_j \in R,\ \lambda_j \geqslant 0,\ |\lambda_j - \lambda| \geqslant 1\right\}.$$

Then there exist positive constants K_1 *and* K_2, *depending only on* c *and* N, *such that*

$$K_1(\lambda+1)^{-N-1/p} \leqslant d_{c,p}(x^\lambda, N) \leqslant K_2(\lambda+1)^{-N-1/p}. \tag{1.8}$$

Under more restrictive conditions (1.8) was proved by Borosh, Chui, and Smith [3].

Two results, closely connected to Theorem 1′, should be pointed out.

Saff and Varga have proved the following theorem (part one of Theorem 3.1 in [4]).

Let the $k+1$ integers $\mu_0, \ldots, \mu_k$ be fixed, with $0 \leqslant \mu_0 < \mu_1 < \cdots < \mu_k$. For each non-negative integer n, set

$$E_n = E_n(\mu_0, \ldots, \mu_k, q) := \inf\left\|x^n\left(x^{\mu_k} - \sum_{j=0}^{k-1} c_j x^{\mu_j}\right)\right\|_{L_q(0,1)}$$

where the infimum is taken over all $(c_0, c_1, \ldots, c_{k-1}) \in R^k$, and where $1 \leqslant q \leqslant \infty$. Then

$$\lim_{n\to\infty} n^{k+1/q} E_n = \frac{\varepsilon_k}{k!} \prod_{j=0}^{k-1} (\mu_k - \mu_j)$$

where

$$\varepsilon_k = \varepsilon_k(q) := \inf\{\|e^{-t}(t^k - h(t))\|_{L_q(0,+\infty)} : h \in \pi_{k-1}\}$$

and π_n is the class of all polynomials of degree at most n.

In [2] one of the authors, using a theorem of Smith [5], has shown that the equality

$$\inf\left\{\left\|x^\lambda - \sum_{j=1}^{l} a_j x^{\lambda_j}\right\|_{L_\infty(0,1)} : a_j \in R,\ \lambda_j \in R,\ |\lambda_j - \lambda| \geqslant 1,\ j = 1, \ldots, l\right\}$$
$$= \inf\{\|x^\lambda(1 - xP(\log x))\|_{L_\infty(0,1)} : P \text{ polynomial of degree } \leqslant l-1\}$$

holds if $\lambda < 1$, and has conjectured that the same equality holds also for $\lambda \geqslant 1$.

2. Proof of Theorem 1.

We shall consider first the case $p=\infty$. Let $M=\max_{c\leqslant x\leqslant 1}|P(x)|$, $0\leqslant c<1$, let $r\in(c,1)$ and $\alpha=r^{1/(m-1)}$. Evaluating the polynomial P at m points α^j, $j=0,1,\ldots,m-1$ we obtain a system of m linear equations

$$\sum_{k=1}^{m} a_{n_k}\alpha^{jn_k}=P(\alpha^j), \qquad j=0,1,\ldots,m-1. \tag{2.1}$$

The determinant of this system is the Vandermonde determinant

$$\Delta_m=\Delta_m(\alpha^{n_1},\alpha^{n_2},\ldots,\alpha^{n_m})=\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \alpha^{n_1} & \alpha^{n_2} & \cdots & \alpha^{n_m} \\ \alpha^{2n_1} & \alpha^{2n_2} & \cdots & \alpha^{2n_m} \\ \vdots & \vdots & & \vdots \\ \alpha^{(m-1)n_1} & \alpha^{(m-1)n_2} & \cdots & \alpha^{(m-1)n_m} \end{vmatrix} \tag{2.2}$$

(2.2)

and so

$$\Delta_m=\prod_{\substack{i,j=1,\ldots,m\\ i<j}}(\alpha^{n_i}-\alpha^{n_j}). \tag{2.3}$$

If $A_{j,k}$ denotes the minor of Δ_m corresponding to the entry in the jth row and the kth column, we obtain from (2.1) that

$$\alpha_{n_k}=\frac{1}{\Delta_m}\sum_{j=1}^{m}(-1)^{j+k}P(\alpha^{j-1})\,A_{j,k}, \qquad k=1,\ldots,m. \tag{2.4}$$

Explicit expressions for the minor $A_{j,k}$ are known, and can easily be deduced. If α^{n_k} is replaced by z, from (2.2) and (2.3) it follows that

$$\sum_{j=1}^{m}(-1)^{j+k}z^{j-1}A_{j,k}$$
$$=(\alpha^{n_1}-z)\cdots(\alpha^{n_{k-1}}-z)(z-\alpha^{n_{k+1}})\cdots(z-\alpha^{n_m})\,\Delta_{m-1,k}$$

where

$$\Delta_{m-1,k}=\Delta_{m-1}(\alpha^{n_1},\ldots,\alpha^{n_{k-1}},\alpha^{n_{k+1}},\ldots,\alpha^{n_m}).$$

Computing the coefficient of z^{j-1} on the right-hand side we find that

$$(-1)^{j+k}A_{j,k}=(-1)^{k-1}\Delta_{m-1,k}(-1)^{m-j}\sigma_{m-j}(\alpha^{n_1},\ldots,\alpha^{n_{k-1}},\alpha^{n_{k+1}},\ldots,\alpha^{n_m}). \tag{2.5}$$

Here σ_q, $q=0, 1,..., m-1$ are the elementary symmetric functions in $m-1$ variables:

$$\sigma_0 = 1 \qquad \text{and} \qquad \sigma_q(y_1,..., y_{m-1}) = \sum y_{s_1} y_{s_2} \cdots y_{s_q}$$

where the summation is extended over all subsets $(s_1, s_2,..., s_q)$ of cardinality q of the set $(1, 2,..., m-1)$.

Combining (2.4) and (2.5) we obtain

$$a_{n_k} = (-1)^{k+m} \frac{\Delta_{m-1,k}}{\Delta_m} \sum_{j=1}^{m} (-1)^{j-1} \times \sigma_{m-j}(\alpha^{n_1},..., \alpha^{n_{k-1}}, \alpha^{n_{k+1}},..., \alpha^{n_m})\, P(\alpha^{j-1}). \tag{2.6}$$

We observe next that $|P(\alpha^{j-1})| \leqslant M$ and that the quantities σ_{m-j} are non-negative. Since $\sum \sigma_{m-j}(y_1,..., y_{m-1}) = \prod (1+y_j)$, we obtain

$$\begin{aligned} &\left|\sum_{j=1}^{m} (-1)^{j-1} \sigma_{m-j}(\alpha^{n_1},..., \alpha^{n_{k-1}}, \alpha^{n_{k+1}},..., \alpha^{n_m})\, P(\alpha^{j-1})\right| \\ &\quad\leqslant M \sum_{j=1}^{m} \sigma_{m-j}(\alpha^{n_1},..., \alpha^{n_{k-1}}, \alpha^{n_{k+1}},..., \alpha^{n_m}) \\ &\quad\leqslant M \prod_{\substack{1 \leqslant j \leqslant m \\ j \neq k}} (1+\alpha^{n_j}) \\ &\quad\leqslant 2^{m-1} M. \end{aligned}$$

On the other hand, from (2.3) it follows that

$$\left|\frac{\Delta_m}{\Delta_{m-1,k}}\right| = (\alpha^{n_1} - \alpha^{n_k}) \cdots (\alpha^{n_{k-1}} - \alpha^{n_k})(\alpha^{n_k} - \alpha^{n_{k+1}}) \cdots (\alpha^{n_k} - \alpha^{n_m}).$$

If $j<k$ we have $\alpha^{n_j} - \alpha^{n_k} = \alpha^{n_j}(1-\alpha^{n_k - n_j}) > \alpha^{n_j}(1-\alpha)$; and if $j>k$, $\alpha^{n_k} - \alpha^{n_j} > \alpha^{n_k}(1-\alpha)$. So

$$\begin{aligned} \left|\frac{\Delta_m}{\Delta_{m-1,k}}\right| &\geqslant (1-\alpha)^{m-1} \alpha^{n_1 + \cdots + n_{k-1} + (m-k)n_k} \\ &\geqslant (1-\alpha)^{m-1} \alpha^{(m-1)n_k}. \end{aligned} \tag{2.8}$$

It follows from (2.6), (2.7), and (2.8) that

$$|a_n| \leqslant \left(\frac{2}{(1-\alpha)\,\alpha^n}\right)^{m-1} M, \qquad \text{where} \quad n = n_k. \tag{2.9}$$

Since $\alpha^{m-1} = r$, and r was an arbitrary number in $[c, 1)$ we can choose now α in $[c^{1/(m-1)}, 1)$ so that the right-hand side of (2.9) takes its smallest value. For that we need to distinguish two cases:

(i) if $n/(n+1) \geqslant c^{1/(m-1)}$, we choose $\alpha = n/(n+1)$;

(ii) if $n/(n+1) < c^{1/(m-1)}$, we choose $\alpha = c^{1/(m-1)}$.

In both cases we have

$$\frac{1}{\alpha^n} \leqslant \left(1 + \frac{1}{n}\right)^n \leqslant e.$$

In the first case we obtain

$$|a_n| \leqslant (2e(n+1))^{m-1} M,$$

in the second case

$$|a_n| \leqslant \left(\frac{2e}{1 - c^{1/(m-1)}}\right)^{m-1} M.$$

Hence we have

$$|a_n| \leqslant A(n+1)^{m-1} \|P\|_\infty \tag{2.10}$$

where $\|P\|_\infty = \max_{c \leqslant x \leqslant 1} |P(x)|$ and

$$A = \left(\frac{2e}{1 - c^{1/(m-1)}}\right)^{m-1}.$$

We consider now the case $1 \leqslant p < \infty$. Let s be a non-negative integer and let

$$Q(x) = -\int_x^1 P(t)\, t^s\, dt.$$

Clearly,

$$Q(x) = \sum_{k=1}^{m} \frac{a_{n_k}}{n_k + s + 1} x^{n_k + s + 1} - C.$$

Thus 0 is a polynomial of length $\leqslant m+1$ and $Q + C$ is a polynomial of length $\leqslant m$. We shall apply the inequality (2.10) twice: first to the polynomial Q to estimate the constant term C, then to the polynomial $Q(x) + C$ to estimate all the coefficients $a_{n_k}/(n_k + s + 1)$.

First application of inequality (2.10) gives

$$|C| \leqslant A\|Q\|_{\infty}. \tag{2.11}$$

Second application gives

$$|a_{n_k}|/(n_k+s+1) \leqslant A(n_k+s+2)^{m-1}\|Q+C\|_{\infty}. \tag{2.12}$$

From (2.11) and (2.12) we deduce

$$|a_{n_k}| \leqslant B(n_k+s+2)^m\|Q\|_{\infty} \tag{2.13}$$

where we can take $B=A(A+1)$. Since

$$Q(x) = -\int_x^1 P(t)\, t^s\, ds,$$

we have

$$\|Q\|_{\infty} \leqslant \int_c^1 |P(t)|\, t^s\, dt$$
$$\leqslant \left(\int_0^1 t^{qs}\, dt\right)^{1/q} \|P\|_{L_p(c,1)}$$
$$\leqslant \left(\frac{1}{sq+1}\right)^{1/q} \|P\|_{L_p(c,1)}$$

or

$$\|Q\|_{\infty} \leqslant \left(\frac{1}{s+1}\right)^{1/q} \|P\|_{L_p(c,1)}$$

where $1/p+1/q=1$. From this inequality and (2.13) follows that

$$|a_{n_k}| \leqslant B(n_k+s+2)^m \frac{1}{(s+1)^{1/q}} \|P\|_{L_p(c,1)}.$$

Choosing here $s=n_k$ we obtain

$$|a_{n_k}| \leqslant 2^m B(n_k+1)^{m-1/q}\|P\|_{L_p(c,1)}$$

and the theorem is proved.

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